Quantum Mechanics:

Description of quantum systems and dynamics

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- Hamilton's equations of motion

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

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- H is *generator* of time translations

$$\delta f = \delta t\{f, H\}$$

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- System evolves according to Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle$$

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$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{H},\hat{f}] + \frac{\partial\hat{f}}{\partial t}$$

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- *Looks* more familiar from classical, c.f., $\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t}$

Aside: Canonical Quantisation

Canonical Quantisation *postulate*

- "Promote" observables to operators:

$$x_i(t),\ p_i(t),\ f_i(t),\ \dots \quad \to \quad \hat{x}_i(t),\ \hat{p}_i(t),\ \hat{f}_i(t),\ \dots$$

- Commutation relations related to Poisson Bracket:

$$[\hat{f}, \hat{g}] = i\hbar\,\{f, g\}$$

	Classical	Quantum
Canonical relations:	$\{x_i, p_j\} = \delta_{ij}$	$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$
Time translation:	$\delta f = \delta t\,\{f, H\}$	$\delta\hat{f} = \frac{i\delta t}{\hbar}\,[H, \hat{f}]$

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- Expanding:

$$\frac{|\psi(t+\delta t)\rangle - |\psi(t)\rangle}{\delta t} = \hat{G}|\psi(t)\rangle \qquad \text{c.f.} \qquad \hat{H}|\psi\rangle = i\hbar\frac{d}{dt}|\psi\rangle$$

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$$\implies \hat{G} = \frac{-i}{\hbar}\hat{H}$$

Time evolution for finite Δt

Time evolution operator for larger Δt

- Write as product of N infinitesimal steps:

$$\hat{U}(\Delta t) = \overbrace{\hat{U}(\delta t)\hat{U}(\delta t)\ldots\hat{U}(\delta t)}^{N \text{ times}} = U(\delta t)^N, \quad N = \frac{\Delta t}{\delta t}$$

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- Taking $\delta t \to 0$:

$$\hat{U}(\Delta t) = \lim_{N\to\infty}\left(1 - \frac{i\delta t}{\hbar}\,\hat{H}\right)^N = e^{-i\Delta t\,\hat{H}/\hbar}$$

- Choose $t_i = 0$;

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$$

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Finally: taking derivative recovers Heisenberg's equations

$$\frac{\mathrm{d}\hat{f}_H}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{H}, U^\dagger\hat{f}_S U] = \frac{i}{\hbar}[\hat{H}, \hat{f}_H]$$